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Computing Anticipatory Systems with Incursion and Hyperincursion

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Abstract

An anticipatory system is a system which contains a model of itself and/or of its environment in view of computing its present state as a function of the prediction of the model. With the concepts of incursion and hyperincursion, anticipatory discrete systems can be modelled, simulated and controlled. By definition an incursion, an inclusive or implicit recursion, can be written as:

$$x(t+1) = F [..., x(t-1), x(t), x(t+1),...]$$

where the value of a variable x(t+1) at time t+1 is a function of this variable at past, present and future times. This is an extension of recursion.
Hyperincursion is an incursion with multiple solutions.

For example, chaos in the Pearl-Verhulst map model :	$x(t+1) = a.x(t).[1 - x(t)]$
is controlled by the following anticipatory incursive model :	$x(t+1) = a.x(t).[1 - x(t+1)]$
which corresponds to the differential anticipatory equation :	$dx(t)/dt = a.x(t).[1 - x(t+1)] - x(t)$.

The main part of this paper deals with the discretisation of differential equation systems of linear and non-linear oscillators. The non-linear oscillator is based on the Lotka-Volterra equations model. The discretisation is made by incursion. The incursive discrete equation system gives the same stability condition than the original differential equations without numerical instabilities. The linearisation of the incursive discrete non-linear Lotka-Volterra equation system gives rise to the classical harmonic oscillator. The incursive discretisation of the linear oscillator is similar to define backward and forward discrete derivatives. A generalized complex derivative is then considered and applied to the harmonic oscillator. Non-locality seems to be a property of anticipatory systems. With some mathematical assumption, the Schrödinger quantum equation is derived for a particle in a uniform potential.
Finally an hyperincursive system is given in the case of a neural stack memory.

Keywords: Control of chaos, Quantum systems, Discrete complex derivative, Incursive anticipatory systems, Hyperincursive memory systems.

1 Introduction

Robert Rosen (1985, p. 341), in the famous book, “tentatively defined the concept of an anticipatory system: a system containing a predictive model of itself and/or of its environment, which allows it to state at an instant in accord with the model's predictions pertaining to a later instant... It is well to open our discussion with a recapitulation of the main features of the modelling relation itself, which is by definition the heart of an anticipatory system”

Robert Rosen (1985), in his book, conjectures that adaptation and learning systems in biological processes are anticipatory systems. For him, anticipation is the central difference between living and non-living systems.

Before explaining my understanding of computing anticipatory systems by incursion and hyperincursion, I would like to point out that my mathematical modelling is drastically different from Robert Rosen one.

1.1 Robert Rosen's Interpretation of Anticipatory Systems

Robert Rosen (1985) states on one hand, that the evolution of an anticipatory system S(t) at each time step is driven by the predictive model M(t+1) at a later time. But, on the other hand, Robert Rosen says that the predictive model M is not affected by the system.

With these statements, a finite difference equation system can thus be written as

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), M(t+\Delta t)] \tag{1a}$$

$$\Delta M/\Delta t = [M(t+\Delta t) - M(t)]/\Delta t = G[M(t)] \tag{1b}$$

If M is model of the system itself, then the model is a model of the system itself without the predictive model guiding it. Thus, the predictive model of the system itself is not a true predictive model of the system itself. This leads to a contradiction.

If M is a model of the environment of the system, and if the environment is affected by the system, this leads also to a contradiction: the model is not predictive because the model is not affected by the system which affects the environment.

In conclusion of this short analysis of Robert Rosen approach to anticipatory systems, it must be stated that there is no drastic difference between his approach and the classical theory of control. Indeed, in control theory, the engineer designs the control function of a system as a function of future objectives which guide this system in feedbacking the outputs of the system to modulate its inputs in view of minimising the distance between the actual outputs to the wanted outputs.

1.2 My Interpretation of Anticipatory Systems

With the Robert Rosen definition of anticipatory systems, I propose to define a computing anticipatory system S as a finite difference equation system

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), M(t+\Delta t)] \tag{2a}$$
$$\Delta M/\Delta t = [M(t+\Delta t) - M(t)]/\Delta t = F[S(t), M(t+\Delta t)] \tag{2b}$$

where the future state of the system S and the model M at time $t+\Delta t$ is a function F of this system S at time t and of the model M at a later time step $t+\Delta t$.
If the model is the system itself, then I write M = S and eqs. 2ab reduce to
$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), S(t+\Delta t)] \tag{2c}$$

what I defined as an incursive system, an inclusive or implicit recursive system: the future state of an incursive system $S(t+\Delta t)$ depends on the past and present state(s) of the system ... $S(t-\Delta t)$, $S(t)$ but also on its future state(s) $S(t+\Delta t)$, $S(t+2\Delta t)$, ...
If I replace $S(t+\Delta t)$ in the second member of eq. 2c by the equation 2c itself, I obtain

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), S(t)+\Delta t.F[S(t),S(t+\Delta t)]] \tag{2c'}$$

in which I can replace indefinitely $S(t+\Delta t)$ by the equation itself leading to an infinite series.
I will show in this paper an example where this infinite series is a simple expression depending only on S(t) with the Pearl-Verhulst chaos map.

If the model is a model of the environment E of the system, I write

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), E(t+\Delta t)] \tag{2d}$$
$$\Delta E/\Delta t = [E(t+\Delta t) - E(t)]/\Delta t = G[E(t), S(t)] \tag{2e}$$

which defines also as an incursive system because the future state of the system is a function of its state at the preceding time and of the future state of its environment. Such a system can be transformed to the recursive system

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), E(t)+\Delta t.G[E(t),S(t)]] \tag{2d'}$$
$$\Delta E/\Delta t = [E(t+\Delta t) - E(t)]/\Delta t = G[E(t), S(t)] \tag{2e'}$$

Examples of such systems will be given in this paper with non-linear Lotka-Volterra and linear harmonic oscillators.

If the environment has also a predictive model of the system, I can write

$$\Delta S/\Delta t = [S(t+\Delta t) - S(t)]/\Delta t = F[S(t), E(t+\Delta t)] \tag{2f}$$
$$\Delta E/\Delta t = [E(t+\Delta t) - E(t)]/\Delta t = G[E(t), S(t+\Delta t)] \tag{2g}$$

which is also an incursive system leading to two crossed infinite series.

1.3 Definition of Incursion

Let us first define the concept of incursion, a contraction of inclusive or implicit recursion.
A simple recursion with a function f[x(t), p] of a variable x(t) depending of time t and parameter p, is defined as

$$x(t+1) = f[x(t), p] \tag{3}$$

where the value of the variable at each instant t+1 is a function of the value of this variable at the preceding time step t. From the knowledge of the initial condition x(0) at time t=0, it is possible to compute all the future states of the variable:

$$x(1) = f[x(0), p]$$
$$x(2) = f[x(1), p]$$
$$\ldots$$
$$x(n) = f[x(n-1), p] \tag{3a}$$

An incursion is an inclusive or implicit recursion, an extension of the recursion in the following way:

$$x(t+1) = f[x(t), x(t+1), p] \tag{4}$$

where the value of the variable at each instant t+1 is a function of the value of this variable at the preceding time step t, but also at time t+1. This defines a self-referential system which is an anticipatory system of itself. The function f describing the dynamics of a system contains a model of itself; indeed x(t+1) in the function f can be replaced in the following way:

$$x(t+1) = f[x(t), f[x(t), x(t+1)], p), p] \tag{4a}$$

where the system explicitly contains a predictive model of itself. Let us give a simple example of such an incursive anticipatory system.

1.4 Control of Chaos in an Incursive Anticipatory System

In population dynamics, the differential equation of the growth of a population following Pear-Verhulst is given by

$$dx(t)/dt = a.x(t).[1-x(t)]-b.x(t) \tag{5}$$

where a is the birth rate and b the death rate and [1-x(t)] represents a control of the birth rate of the population. Indeed, without control, the population would grow to infinity if $a > b$ and to zero if $a < b$.
A finite difference Pearl-Verhulst equation is given by:

$$x(t+1) = x(t) + a.x(t).(1-x(t)) - b.x(t) \tag{5a}$$

where x(t) is the value of the population at time t and a is the growth rate and b the decay rate.

In taking b=1, the well-known chaos map, given in Fig. 1a, is obtained:

$$x(t+1) = a.x(t).(1-x(t)) \quad (5a')$$

For value of a = 4, the system shows chaos and the future of this system is unpredictable from its initial condition x(0) at time t = 0. This is the sensibility to initial conditions.

An anticipatory system can be built from this eq. 5a' in writing:

$$x(t+1) = a.x(t).(1-x(t+1)) \quad (6)$$

where now the saturation factor is a predictive function of the population at a later time step.
In replacing x(t+1) in the second member of eq. 6 by eq. 6 itself, we obtain:

$$x(t+1) = a.x(t).(1-a.x(t).(1-x(t+1))) \quad (6a)$$

which is an incursive system. In continuing to replace x(t+1) by itself, an infinite sequence is obtained:

$$x(t+1) = a.x(t).(1-a.x(t).(1- a.x(t).(1- a.x(t).(1- a.x(t).(1- a.x(t).(1- a.x(t).(1-...))))))) \quad (6b)$$

There is a paradox which in fact is easy to understand. If an anticipatory system contains a model of itself, this means that the model of itself must include also the model of itself and so on until infinity. There are an infinity of embedded models in each other. This looks without hope to resolve the problem or to simulate such systems. Mathematically, there is a means to resolve this paradox. Indeed, eq. 6b can be written as

$$x(t+1) = a.x(t).[1-a.x(t)+a^2.x(t)^2-a^3.x(t)^3+...] = a.x(t) / [1+a.x(t)] \quad (6c)$$

because

$$[1-a.x(t)+a^2.x(t)^2-a^3.x(t)^3+...] = \sum_{n=0}^{\infty} [(-a.x(t)]^n = 1 / [1+a.x(t)] \quad (6c')$$

This is an important result which shows that the future value of a variable x at time t+1 can be represented by an infinite sum of this variable x at the present time x(t) which converges to a simple function depending on this variable x(t) at the current present time.

An anticipatory system can be related to a hyperset H defined as a collection of its elements E and itself: H ={E,H}.

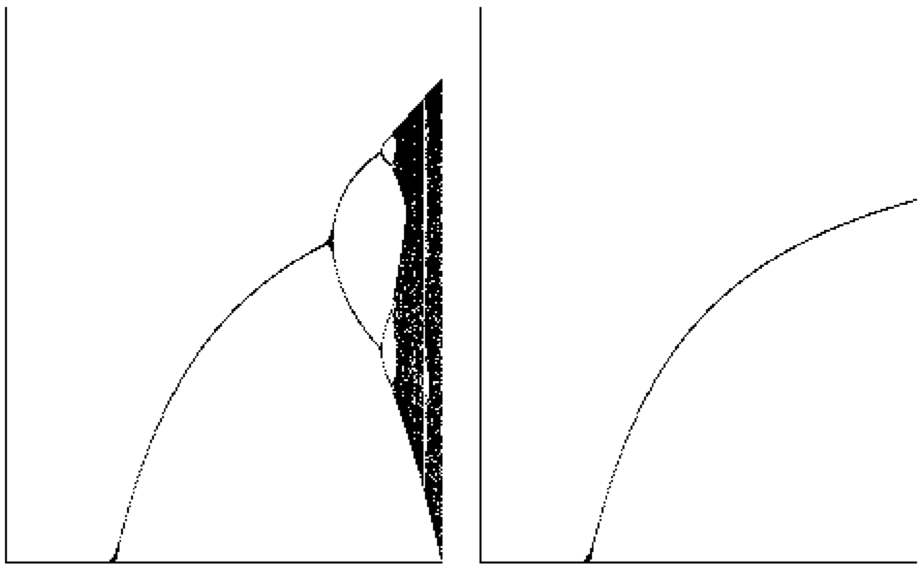
Jon Barwise and Larry Moss (1991) introduced this concept of hyperset for defining infinite descending sequences. They demonstrate that such equations are ruled out by an accepted axiom of set theory, the Axiom of Foundation.

Peter Aczel (1988) work has taught that the axiom of foundation is not a necessary part of a clear picture of the universe of sets. The Antifoundation Axiom was thus proposed (see Jon Barwise and Larry Moss, 1991, for a very interesting development of new directions of research in hypersets theory). In a simple manner, it is established that a classical set cannot be a member of itself. So, self-referential systems have today no theoretical well-established framework.

The incursive equation of the anticipatory system 6 can be thus written as the recursive anticipatory system

$$x(t+1) = a.x(t).[1-a.x(t)/[1+a.x(t)]] \quad (6d)$$

With such an anticipation, eq. 6d doesn't show chaos. This anticipatory system has two fixed points which represent its implicit finality or teleonomy. The goal or objective of this anticipatory system is not explicitly imposed from outside the system like in control theory but is determined by the stability or instability points of the system itself.



Figures 1ab: (a) Bifurcation diagram of eq. 5a', x(t) as a function of a.
(b) Incursive anticipatory control of chaos with eq.6c, x(t) as a function of a.

Eq. 6d can be transformed to the following differential equation

$$dx/dt = a.x(t). [1-a.x(t)/[1+a.x(t)]] - x(t) = a.x(t)/[1 + a.x(t)] - x(t) \quad (6e)$$

Let us show that this is actually an anticipatory system in considering the following anticipatory differential equation

$$dx(t)/dt = a.x(t).[1 - x(t + 1)] - x(t) \quad (6f)$$

which is an anticipatory system because the derivative of x(t) depends on x(t) but also on x(t+1) at a future time. From the discretisation formula of the time derivative, we can write

$$x(t+1) = x(t) + dx(t)/dt \quad (6g)$$

and in replacing this equation 6g in eq. 6f, we obtain

$$dx(t)/dt = [a.x(t).[1 - x(t)] - x(t)]\ ]/[1 + a.x(t)] = a.x(t)/[1 + a.x(t)\ ] - x(t) \quad (6h)$$

which is identical to eq. 6e. This equation looks similar to the Monod-Michaelis-Menten equation largely used in modelling population dynamics, growth of bacteria, biochemical reactions and economics etc.

Let us show now an hyperincursive anticipatory system.

1.5 Definition of an Hyperincursive Anticipatory System

The following equation

$$x(t) = a.x(t+1).(1-x(t+1)) \quad (7)$$

defines an hyperincursive anticipatory system. Hyperincursion is an incursion with multiple solutions. With a = 4, mathematically x(t+1) can be defined as a function of x(t)

$$x(t+1) = 1/2 \pm 1/2 \sqrt{(1-x(t))} \quad (8)$$

where each iterate x(t) generates at each time step two different iterates x(t+1) depending of the plus minus sign. The number of future values of x(t) increases as a power of 2. As the system can only take one value at each time step, something new must be added for resolving the problem.
Thus, the following decision function u(t) can be added for making a choice at each time step:

$$u(t) = 2.d(t)-1 \quad (9)$$

where u = +1 for the decision d = 1 (true) and u = -1 for the decision d = 0 (false).
In introducing eq. 9 to eq. 8, the following equation is obtained:

$$x(t+1) = 1/2 + (d(t)-1/2). \sqrt{(1-x(t))} \quad (10)$$

The decision process could be explicitly related to objectives to be reached by the state variable x of this system. This is important to point out that the decisions d(t) do not influence the dynamics of x(t) but only guide the system which creates itself the potential futures.
This hyperincursive anticipatory system was proposed as a model of a stack memory in neural networks (Dubois, 1996c) which is developed at the end of this paper.

2 Incursive Anticipatory Linear And Non-Linear Oscillators

In computer science, the discretisation of differential equations systems gives sometimes numerical instabilities which can be controlled by incursive anticipatory discretisation.
Let us first consider the non-linear model given by the discretised Lotka-Volterra equations:

$$X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)] \tag{13a}$$
$$Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.X(t).Y(t)] \tag{13b}$$

where t is a discrete time with steps Δt, and a, b, c, d are the parameters.
Analytical solutions exist only for small oscillations from the steady state X_0= c/d and Y_0=a/b, which are identical to a harmonic linear oscillator. In taking $X=X_0 + x$ and $Y=Y_0 + y$, when x and y are small, the linearisation of these equations gives

$$x(t+\Delta t) = x(t) - \Delta t.(bc/d).y(t) \tag{13a'}$$
$$y(t+\Delta t) = y(t) + \Delta t.(ad/b).x(t) \tag{13b'}$$

With the change in variables q(t) = x(t), p(t) = - y(t) and in taking bc/d = 1/m and ω^2 = ac/m, the linear harmonic oscillator equations system is obtained:

$$q(t+\Delta t) - q(t) = \Delta t.p(t)/m \tag{14a}$$
$$p(t+\Delta t) - p(t) = - \Delta t.m.\ \omega^2.q(t) \tag{14b}$$

in defining by q the position and p the momentum (p = m.v where m is the mass and v the velocity), and where ω is the pulsation. These equation can be reduced to an equation in q(t):

$$q(t+2\Delta t) - 2q(t+\Delta t) + q(t) = - \Delta t^2.\ \omega^2.q(t) \tag{14c}$$

which corresponds to a discretisation of the second derivative of q(t+Δt).
But the solutions all these equations are unstable.

The following Fig. 2 shows the instability of the Lotka-Volterra discrete model.

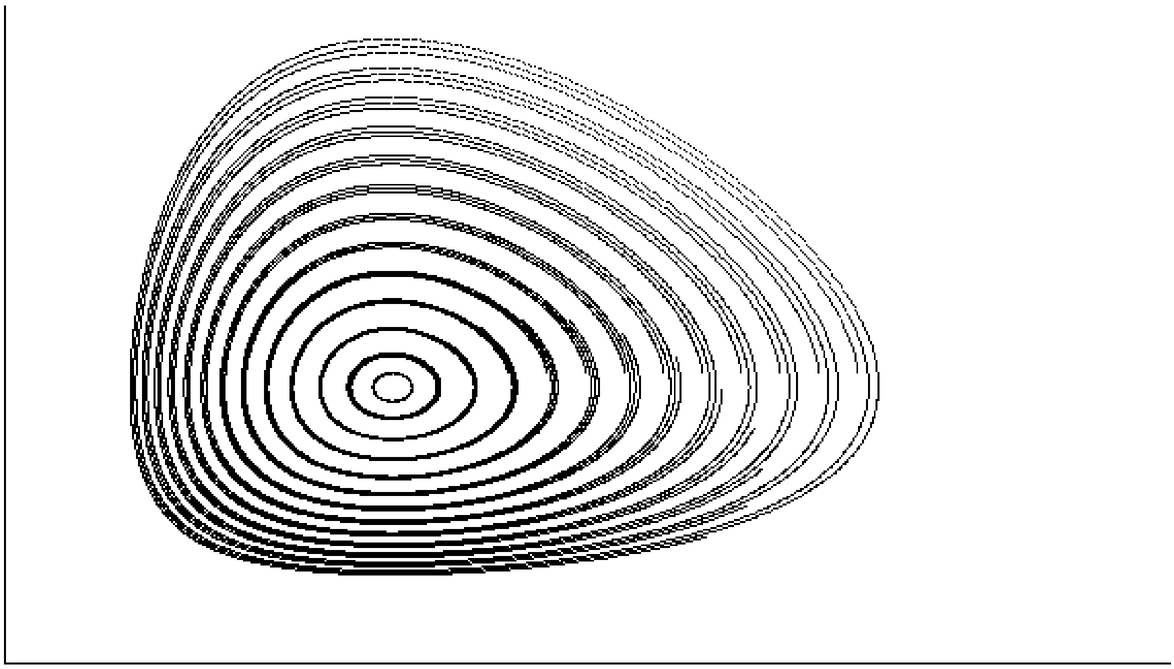


Figure 2: Simulation of eqs. 13ab, Y(t) as a function of X(t), with different initial conditions. This discretisation of the Lotka-Volterra model is unstable.

Different incursive discrete equations systems exist to control the oscillations of these non-linear and linear discrete equations.
The iterative values of $X(t+\Delta t)$ of the first equation (13a) can be propagated to the second equation (13b), in an incursive way, as proposed by Dubois [1992, 1993]:

$$X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).Y(t)] \quad (15a)$$
$$Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.\mathbf{X(t+Dt).}Y(t)] \quad (15b)$$

The simulation gives solutions with orbital stability (see Fig. 3a) and chaos (see Fig. 3b) depending on the values of the parameters [Dubois, 1992].

The linearised incursive Lotka-Volterra equations give the incursive harmonic oscillator ones:

$$q(t+\Delta t) = q(t) + \Delta t.p(t)/m \quad (15a')$$
$$p(t+\Delta t) = p(t) - \Delta t.m.\ \omega^2.q(t+\Delta t) \quad (15b')$$

with orbital stability [Dubois, 1995). In replacing $q(t+\Delta t)$ in eq. 15b' by eq. 15a', eq. 15b' becomes

$$p(t+\Delta t) = p(t) - \Delta t.m.\ \omega^2.q(t) - \Delta t^2.\ \omega^2.P(t) \quad (15b'')$$

where the term in Δt^2 disappears when Δt tends to zero. Eqs. 15a'b' can be reduced to an equation in q(t):

$$q(t+\Delta t) - 2q(t) + q(t-\Delta t) = - \Delta t^2 . \omega^2 .q(t) \quad (15c')$$

which is time invertible in replacing Δt by $- \Delta t$. With a time translation of $- \Delta t$, eq. 15b' writes

$$p(t) = p(t - \Delta t) - \Delta t.m. \omega^2 .q(t) \quad (15d')$$

which corresponds to a backward derivative of the momentum. In fact two derivatives can be defined for a discrete variable x:

$$\Delta_f x / \Delta t = (x(t + \Delta t) - x(t))/\Delta t \quad (16)$$
$$\Delta_b x / \Delta t = (x(t) - x(t - \Delta t))/\Delta t \quad (17)$$

The forward derivative 16 and the backward derivative 17 are not always equal (only at the limit for $\Delta t=0$ for continuous derivable equations); for non-derivable continuous equations like in fractal equations systems, two derivatives must be defined. Let us remark that when Δt is replaced by $- \Delta t$, the forward and backward derivatives 16 and 17 becomes the backward and forward ones. Moreover, the successive application of the forward derivative to the backward derivative, or the inverse, gives the second order derivative, which is time invertible:

$$\Delta^2 x / \Delta t^2 = x(t + \Delta t) - 2x(t) + x(t - \Delta t)] / \Delta t^2$$

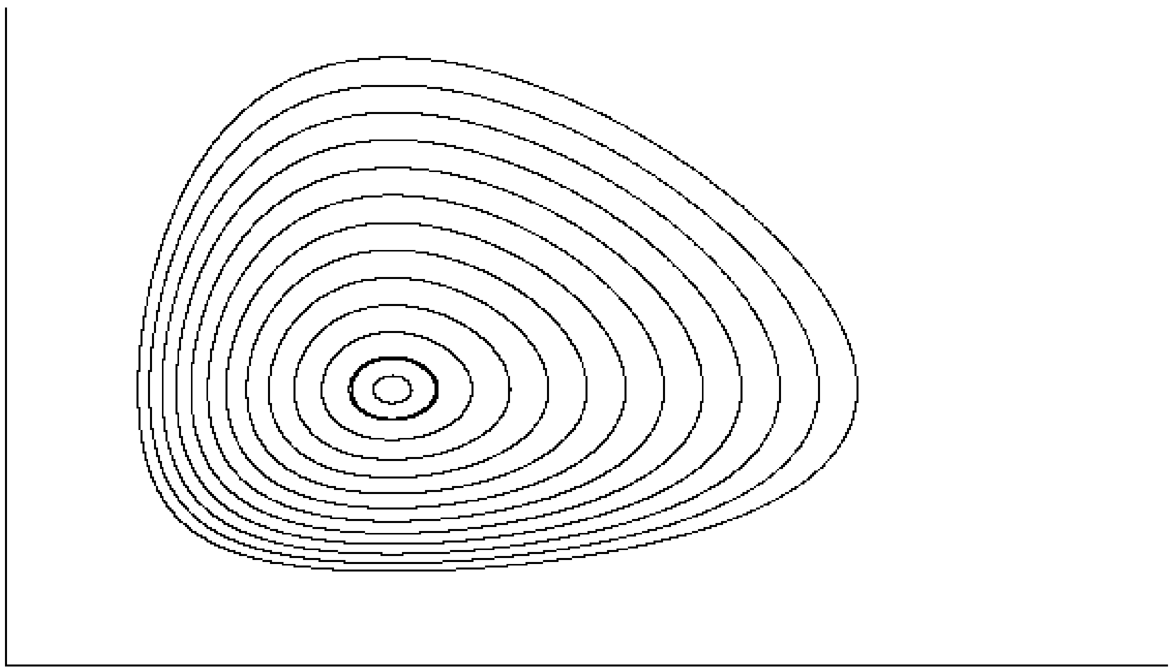


Figure 3a: Orbital stability of the incursive anticipatory Lotka-Volterra discrete eqs. 15 ab in the phase space, Y(t) as a function of X(t).

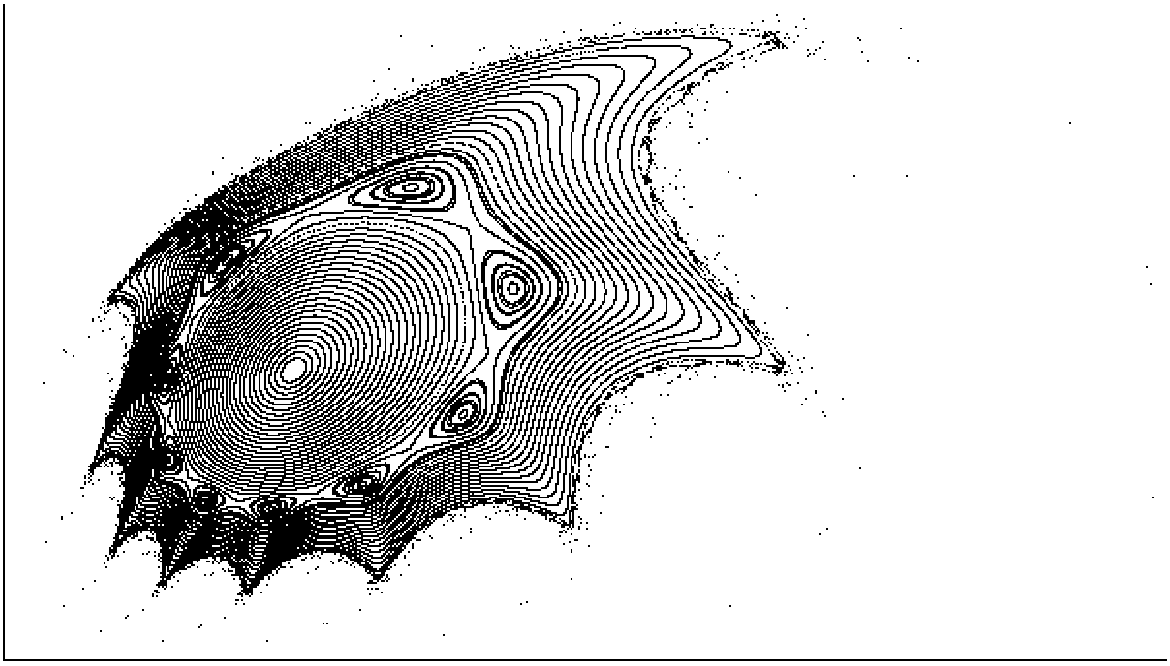


Figure 3b: Orbital stability and chaos of the incursive anticipatory Lotka-Volterra discrete eqs. 15 ab in the phase space, Y(t) as a function of X(t).

2.1 A generalized Discrete Derivative with Forward and Backward derivatives

Let us define a generalized discrete derivative by a weighted sum of these derivatives as follows [Dubois, 1995]:

$$\Delta_w x/\Delta t = w.\Delta_f x/\Delta t+(1-w).\Delta_b x/\Delta t = [w.x(t+\Delta t) + (1-2.w).x(t)+(w-1).x(t-\Delta t)]/\Delta t \quad (18)$$

where the weight w is defined in the interval 0,1. For w = 1, the forward derivative 16 is obtained and for w = 0, the backward derivative 17. For w = 1/2, derivative 18 becomes

$$\Delta_{1/2} x/\Delta t = (x(t+\Delta t)-x(t-\Delta t))/2\Delta t=[\Delta_f x/\Delta t + \Delta_b x/\Delta t]/2 \quad (18a)$$

which is an average derivative. With this generalized derivative 18, the discrete harmonic oscillator equations system can be defined by:

$$(1-w).q(t+\Delta t)+(2w-1).q(t)-w.q(t-\Delta t) = \Delta t.p(t)/m \quad (19a)$$
$$w.p(t+\Delta t)+(1-2w).p(t)+(w-1).p(t-\Delta t) = - \Delta t.\omega^2.m.q(t) \quad (19b)$$

which is identical to the equations 15a',b' for w = 0.

For w = 1, the equations 19a,b correspond to the linearised incursive Lotka-Volterra equations:

$$Y(t+\Delta t) = Y(t) + \Delta t.[-c.Y(t) + d.X(t).Y(t)] \quad (20a)$$
$$X(t+\Delta t) = X(t) + \Delta t.[a.X(t) - b.X(t).\mathbf{Y(t+Dt)}] \quad (20b)$$

where now we propagate the value of $Y(t+\Delta t)$ in the equation of $X(t+\Delta t)$.

The simulation gives solutions with orbital stability (see Fig. 4a) and chaos (see Fig. 4b) depending on the values of the parameters [Dubois, 1992].

For w = 1/2, the equations 19a,b become:

$$q(t+\Delta t) = q(t-\Delta t) + 2.\Delta t.p(t)/m \quad (21a)$$
$$p(t+\Delta t) = p(t-\Delta t) - 2.\ \Delta t.\ \omega^2.m.q(t) \quad (21b)$$

which are symmetrical, but 4 initial conditions are to be defined instead of two. These are similar to the following linearised discrete Lotka-Volterra equations

$$X(t+\Delta t) = X(t-\Delta t) + 2.\Delta t.[a.X(t) - b.X(t).Y(t)] \quad (22a)$$
$$Y(t+\Delta t) = Y(t-\Delta t) + 2.\Delta t.[-c.Y(t) + d.X(t).Y(t)] \quad (22b)$$

for which 4 initial conditions are to be defined.

The simulation in Figure 5a gives solutions with different initial conditions as in Figs. 3a and 4a. For an other set of parameters Figs. 4bcdefghi show chaos with different initial conditions.

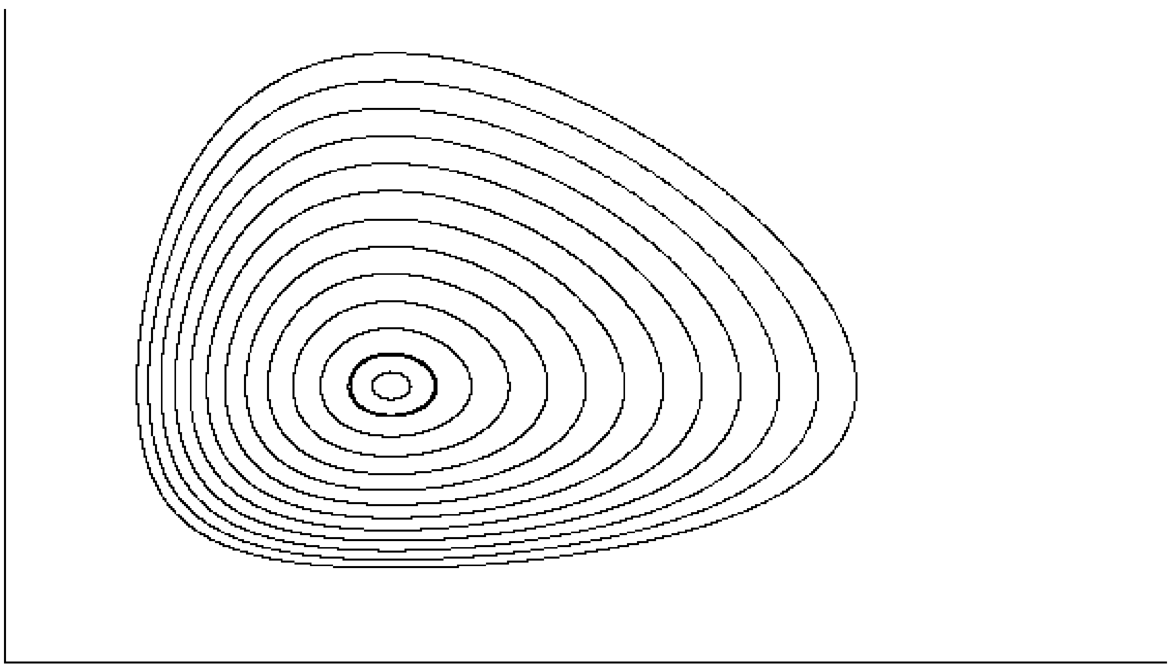


Figure 4a: Orbital stability of the second incursive anticipatory Lotka-Volterra discrete eqs. 20 ab in the phase space, Y(t) as a function of X(t).

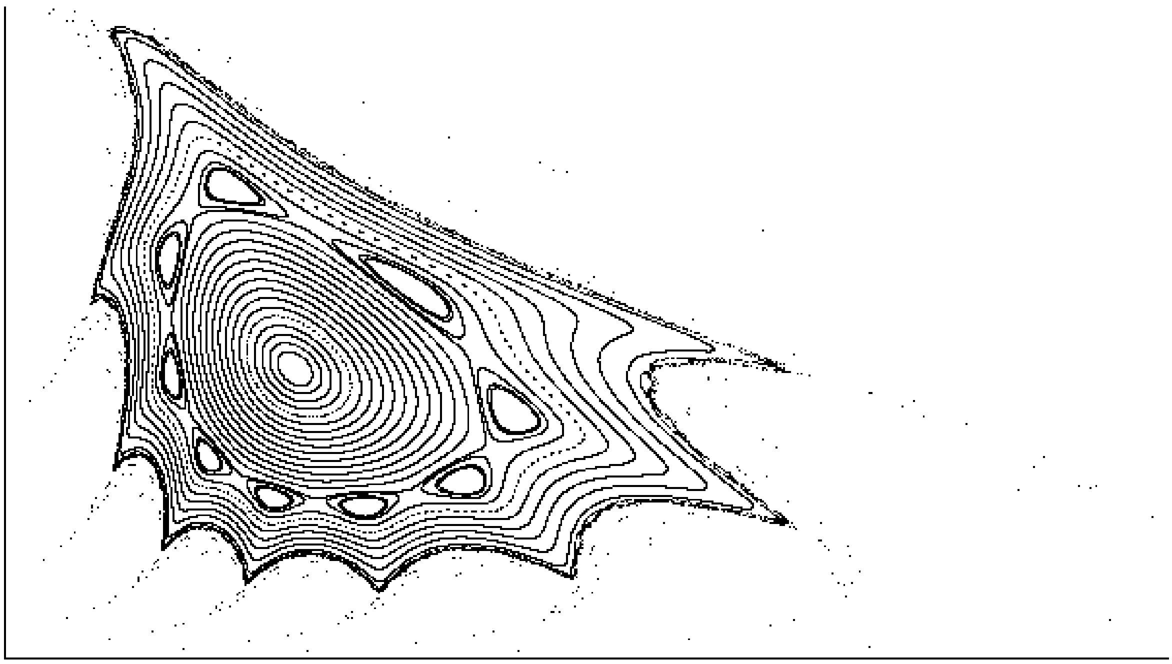


Figure 4a: Orbital stability and chaos of the second incursive anticipatory Lotka-Volterra discrete eqs. 20 ab in the phase space, Y(t) as a function of X(t).

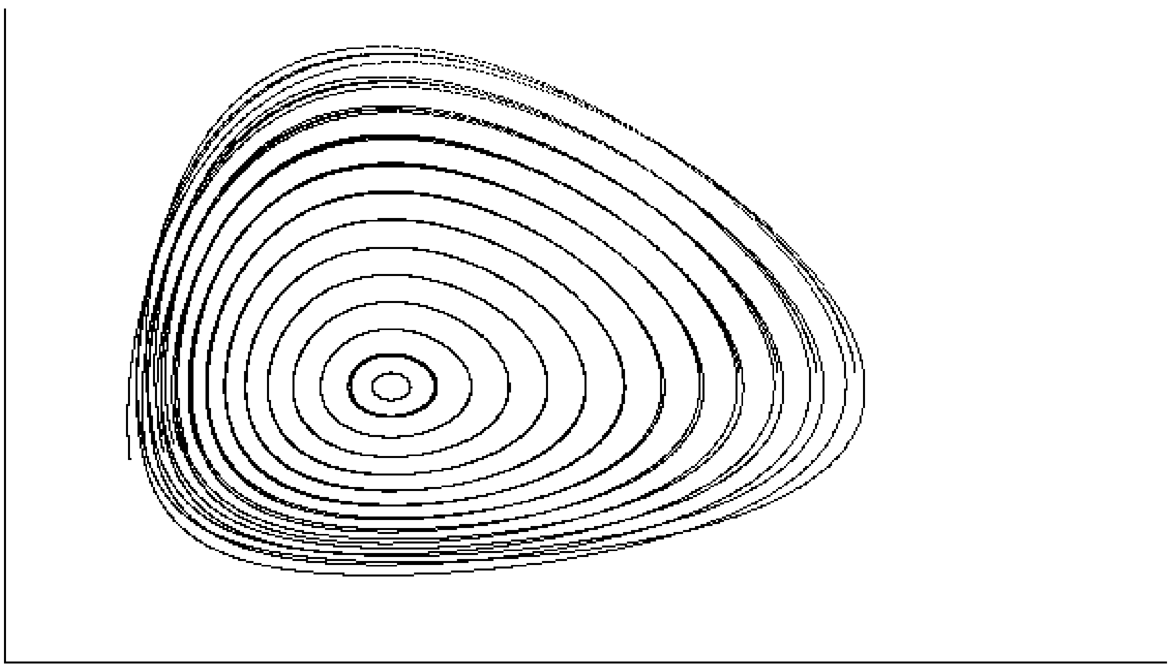


Figure 5a: The simulation of eqs. 22ab gives solutions in the phase space, Y(t) as a function of X(t), with different initial conditions as in Figs. 3a and 4a.

3 A Generalized Complex Discrete Derivative

From eq.8 of the generalized discrete derivative, the second order derivative is given by the successive application of eq.8 for w and (1-w), or the inverse:

$$\Delta_w \Delta_{1-w} x/\Delta t^2 =$$
$$[x(t+\Delta t)-2x(t)+x(t-\Delta t)]/\Delta t^2 + w(1-w)[x(t+2\Delta t)-4x(t+\Delta t)+6x(t)-4x(t-\Delta t)+x(t-2\Delta t)]/\Delta t^2$$
$$= \Delta_{1-w}\Delta_w x/\Delta t^2 \qquad (23)$$

which is the sum of the classical discrete second order derivative and a factor, weighted by w(1 - w), which is similar to a fourth order discrete derivative (multiplied by Δt^2). For w = 0 and w = 1, the classical second order derivative is obtained: w(1 - w) = 0.
For w = 1/2, w(1 – w) = 1/4, the second order derivative is also obtained but with a double time interval 2 Δt.
In choosing the value of the w(1 – w) equal to 1/2, we obtain weights w, solution of

$$w^2 - w + 1/2 = 0 \qquad (24)$$

which are given by the complex numbers

$$w = 1/2 \pm i/2 \qquad (25)$$

and 1-w = 1 / 2 ± (- i / 2) = w^*, where w^* is the complex conjugate of w.

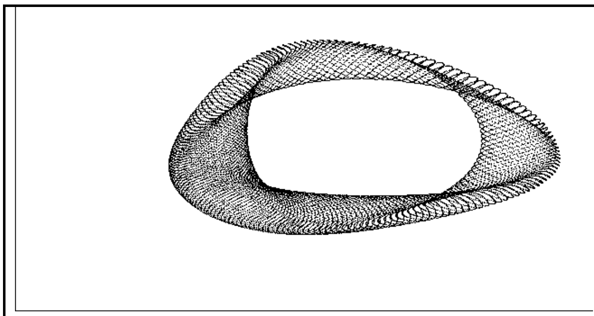


Figure 5b

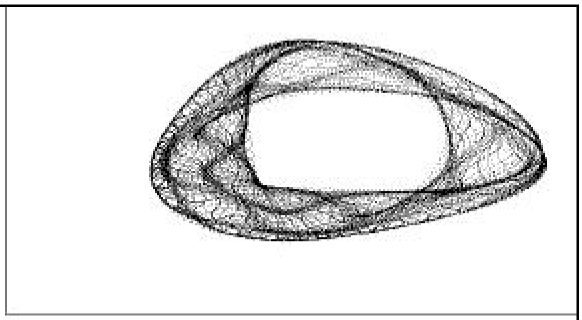


Figure 5c

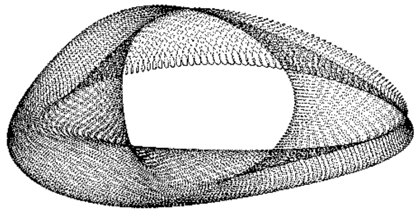
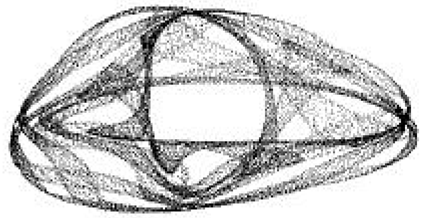
	

Figure 5d	**Figure 5e**

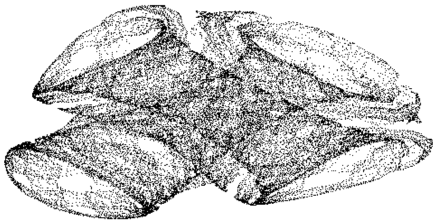
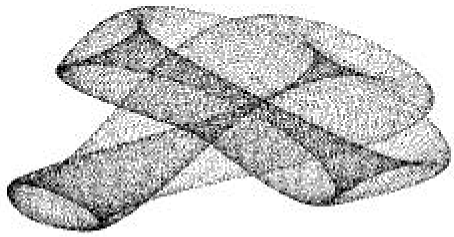

	

Figure 5f	**Figure 5g**

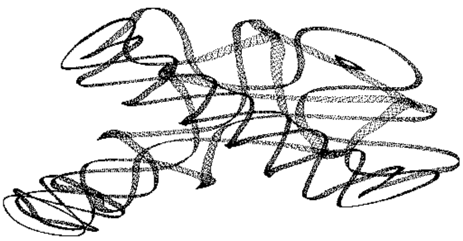
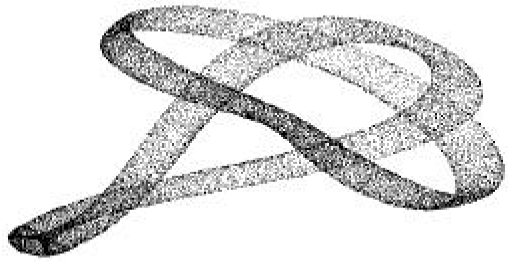

	

Figure 5h	**Figure 5i**

Figures 5bcdefghi: Simulation of eqs. 22ab in the phase space, Y(t) as a function of X(t).

So eq.18 of the generalized discrete derivative can be rewritten as

$$\Delta_w z/\Delta t = w_{\cdot Df}\, z/\Delta t + w^*_{.Db}\, z/\Delta t = [w.z(t+\Delta t)+(w^* - w).z(t)+w^*.z(t-\Delta t)]/\Delta t \qquad (18')$$

where the generalized complex derivative can be applied to a complex variable z = x + i y.

The second order derivative is given by the successive application of eq.18' for w and w^*, or the inverse:

$$\Delta_w \Delta_{w^*} z / \Delta t^2 = [z(t+\Delta t)-2z(t)+z(t-\Delta t)] / \Delta t^2 + w\, w^* [z(t+2\Delta t)-4z(t+\Delta t)+6z(t)-4z(t-\Delta t)+ z(t-2\Delta t)] / \Delta t^2 = \Delta_{w^*} \Delta_w z / \Delta t^2 \qquad (25)$$

which is the sum of the classical discrete second order derivative and a factor, weighted by the real number ww^*, which is similar to a fourth order discrete derivative (multiplied by Δt^2).

3.1 Nottale's Complex Velocity

With the complex weight w given by eq. 15, the first derivative of the position x (eq. 18) gives rise to the complex velocity v

$$v = [\, x(t+\Delta t) - x(t-\Delta t)\,]/2\Delta t \pm i\, [\, x(t+\Delta t) - 2x(t) + x(t-\Delta t)\,]/2\Delta t \qquad (26)$$

In defining a forward velocity

$$v_f = [\, x(t+\Delta t) - x(t)\,]/\Delta t \qquad (27a)$$

and a backward velocity

$$v_b = [\, x(t) - x(t-\Delta t)\,]/\Delta t \qquad (27b)$$

the complex velocity 26 is given by

$$v = [\, v_f + v_b\,]/2 \pm i\, [\, v_f - v_b]/2 \qquad (28)$$

where the real part of the velocity is the average of the forward and backward velocities and the imaginary part is the difference of these forward and backward derivatives. In considering the inverse time interval in replacing Δt by $-\Delta t$, the forward and backward derivatives becomes the backward and forward derivatives. So, if the velocity is time invertible, the plus and minus signs correspond to $\Delta t \leq 0$ and $\Delta t \geq 0$. In the continuous limit Δt tending to zero, $v_f = v_b$, and the classical Newtonian velocity is rediscovered and the imaginary part tends to zero.
The kinetic energy is a real number given by

$$vv^*/2 = [\, v_f^2 + v_b^2\,]/4 \qquad (28a)$$

Let us remark also that the acceleration given by the second derivative of the position x is a real variable, because $w(1 - w) = 1/2$ is real in eq. 23.

The complex velocity given by eq. 28 is similar to the complex velocity proposed by L Nottale [1993]

$$v = [\, v_f + v_b\,]/2 - i\, [\, v_f - v_b]/2 \qquad (29)$$

where only a negative imaginary part is present. In his paper, L. Nottale proved that the definition of backward and forward velocities and the complex energy with the Wiener process and Newtonian law, give the Schrödinger quantum equation.
Let us show now, that without Wiener process and a real energy, the Schrödinger equation can be obtained from the generalized complex discrete derivative applied to a Newtonian harmonic oscillator.

3 Incursive Oscillator Related to Schrödinger's Quantum Equation

It is well-known that the Schrödinger equation is defined as a complex differential equation

$$i \hbar \partial \phi / \partial t = - (\hbar^2 /2m) \partial^2 \phi / \partial s^2 + V \phi \tag{30}$$

in one spatial dimension s for a particle in a potential V(s), where $\phi(s,t)$ is the wave function depending on space s and time t.
A Newtonian particle in a harmonic potential is given by the equation system

$$dx/dt = p/m \tag{31a}$$
$$dp/dt = - m \omega^2 x \tag{31b}$$

In defining a complex function, similar to the Heisenberg formalism,

$$F = (m/2)^{1/2} \omega x + i p/(2m)^{1/2} \tag{31c}$$

we obtain
$dF/dt = (m/2)^{1/2} \omega p/m - i m \omega^2 x /(2m)^{1/2}$
$i dF/dt = (m/2)^{1/2} \omega^2 x + i \omega p/(2m)^{1/2}$
or

$$i dF/dt = \omega F \tag{31d}$$

The generalized complex discrete derivative 26 applied to eq. 31d gives
$i [F(t+\Delta t) – F(t-\Delta t)]/2dt + i [F(t+dt) – 2F(t) + F(t-dt)]/2\Delta t = \omega F(t)$
or, in multiplying both members by $\hbar$,

$$i \hbar [F(t+\Delta t)–F(t-\Delta t)]/2 \Delta t = [-(\hbar/2) [F(t+\Delta t)–2F(t)+F(t-\Delta t)]/\Delta t]+\hbar \omega F \tag{32}$$

The first factor in the second member is similar to a diffusion in time of the position and momentum. This could be interpreted as a temporal non-locality. In taking the real and imaginary part of F = R + iI, we can write eq. 32 as

$$R(t+\Delta t) = R(t-\Delta t) - [I(t+\Delta t)–2I(t)+I(t-\Delta t)] + 2 \Delta t \omega I \tag{32a}$$
$$I(t+\Delta t) = I(t-\Delta t)] + [R(t+\Delta t)–2R(t)+R(t-\Delta t)] - 2 \Delta t \omega R \tag{32b}$$

which is an incursive anticipatory equation system, because $R(t+\Delta t)$ is a function of $I(t+\Delta t)$ and $I(t+\Delta t)$ a function of $R(t+\Delta t)$. The real and imaginary parts of F have a predictive model of each other. Can we conclude that an anticipatory system has the property of temporal non-locality? We will show that eq. 32

can be transformed to a system similar to a quantum system for which spatial non-locality is a well-known property. So a quantum system could be interpreted as a spatial anticipatory system.

In view of obtaining a diffusion in space, let us define the position by $x = x(s, t)$ and the momentum by $p = p(s, t)$, where s is a spatial coordinate, the function $F = F(s, t)$, and eq. 32 becomes:

$$i\hbar\,[F(s,t+\Delta t)-F(s,t-\Delta t)]/2\Delta t = [-(\hbar/2)\,[F(s,t+\Delta t)-2F(s,t)+F(s,t-\Delta t)\,]/\Delta t]+\hbar\,\omega\,F(s,t) \qquad (32d)$$

In inverting the space s and time t variables in the function F(s,t) in the first factor of the second member, the second time derivative is transformed into a second space derivative

$$i\hbar[F(s,t+\Delta t)-F(s,t-\Delta t)]/2\Delta t = [-\,(\hbar\,c^2\,\Delta t\,/2)[F(t,s+\Delta s)-2F(t,s)+F(t,s-\Delta s)\,]/\Delta s^2]+\hbar\omega F \quad (33)$$

We can justify this in reference to the classical wave equation. Indeed, let us consider the one dimension wave equation

$$\partial^2 x(s,t)/\partial t^2 = c^2\,\partial^2 x(s,t)/\partial s^2 \qquad (34a)$$

where x(s,t) is the value of the wave at position s at time t, and c the velocity. This differential equation can be replaced by the finite difference equation

$$x(s,t+\Delta t)-2.x(s,t)+x(s,t-\Delta t) = c^2.\Delta t^2.[x(s+\Delta s,t)-2.x(s,t)+x(s-\Delta s,t)]/\Delta s^2 \qquad (34b)$$

After Finkelstein [1996], the Planck constants: $L_P = 1.6\ 10^{-35}$m, $T_P = 10^{-44}$ s and $M_P = 2.5\ 10^{-8}$ kg can be deduced from the Maxwell constant $c = 3.0\ 10^8$ m/s, the Newton constant $G = 6.67\ 10^{-11}$ Nm^{2}kg and the Plank constant $\hbar = 1.05\ 10^{-34}$ Js.
From these Planck constants, we deduce $M_P c^2 T_P = \hbar$ or $c^2 T_P = \hbar / M_P$
where the Plank interval of time T_P is related to the mass M_P of a particle. So, similarly, we deduce $c^2\,\Delta t = \hbar / m$, where Δt is the interval of time related to the mass m of a particle.
In taking $c^2\,\Delta t = \hbar / m$, eq. 33 becomes

$$i\hbar\,[\,F(s,t+\Delta t)-F(s,t-\Delta t)\,]/2\Delta t = -(\hbar^2/2m)[F(s+\Delta s,t)-2F(s,t)+F(s-\Delta s,t)]/\Delta s^2 + \hbar\omega F(s,t) \quad (35)$$

This discrete equation 35 gives the following finite difference equation

$$i\hbar\,\Delta F/\Delta t = (-\,\hbar^2\,/2m)\,\Delta^2\,F/\,\Delta s^2 + \hbar\,\omega\,F \qquad (36)$$

which is formally similar to the Schrödinger equation for a free particle in a uniform potential
$V = \hbar\,\omega$ along the s axis. Eq. 36 can be split into the real and imaginary parts of F in function of x and v:

$$\Delta x(s,t)\,/\,\Delta t = -(\hbar/2m\omega)\,\Delta^2\,v(s,t)\,/\,\Delta s^2 + v(s,t) \qquad (37a)$$
$$\Delta v(s,t)\,/\,\Delta t = (\hbar\omega/2m)\,\Delta^2\,x(s,t)\,/\,\Delta s^2 - \omega^2\,x \qquad (37b)$$

When the mass becomes higher and higher, the diffusive factors vanish and the Newtonian formalism is obtained where the wave packet tends to a particle.

For ω tending to zero, we will consider the following equation derived from eqs. 36a-b

$$\Delta^2 x(s,t) / \Delta t^2 = -(\hbar^2/4m^2)\, \Delta^4 x(s,t) / \Delta s^4 + (\hbar\, \omega / m)\, \Delta^2 x(s,t) / \Delta s^2 - \omega^2 x \qquad (38)$$

which gives for ω tending to zero

$$\Delta^2 x(s,t) / \Delta t^2 = -(\hbar^2/4m^2)\, \Delta^4 x(s,t) / \Delta s^4 \qquad (39)$$

which is similar to the quantum equation of the real part of the wave function for a free particle. Indeed, in defining the real and imaginary parts of the wave function

$\phi(s,t) = x(s,t) + i\, y(s,t)$

in eq. 30 with V = 0, one obtains

$$\partial^2 x(s,t) / \partial t^2 = -(\hbar^2/4m^2)\, \partial^4 x(s,t) / \partial s^4 \qquad (40)$$

The ratio $\Delta s^2 / \Delta t \propto c^2\, dt = \hbar / m$ is related to the Planck constant. When $\hbar$ becomes smaller and smaller, the Newtonian formalism is obtained [cf. Gutzwiller, 1990].

In fact, the classical Newtonian formalism can be obtained for particles or systems (like molecules) with a high mass m: in this case, the spatial diffusion factor vanishes, which means also that Δs tends to zero with Δt (the fractal dimension is then $D_s = 1$ in the Euclidean space of Newton formalism).

The factor before the space derivative is similar to a diffusion coefficient [cf. Gutzwiller, 1990].

In our framework, we have transformed a time diffusion to a space diffusion to obtain the Schrödinger equation in relation to the wave equation.

When Δs^2 tends to zero with Δt, this means that the space interval has a fractal structure of fractal dimension equal to $D_s = 2$ in agreement with Nottale [1989] conjecture.

An other interpretation is that c^2 tends to infinity with Δt: the velocity becomes infinite: this is not in contradiction with non-relativistic quantum mechanics.

When the velocity of the particle tends to the velocity of light, this formalism is no more available: following Nottale the time has the fractal dimension D = 2. In quantum relativity, $\hbar\omega$ could represent the mass energy of a particle: $\hbar\omega = mc^2$. Research in this direction is in progress.

I have tested on computer the validity of the finite difference eq. 33 in considering as usual by physicists $\hbar$ = 1 and c = 1 with $\hbar\ \omega = 1$, for a particle between two reflecting walls at s = 0 and s = 200 ($\Delta s^2 / \Delta t$ = 1): this is a quantum cellular automata.

The simulations are given in Figures abc and 7.

Fig. 6abc show the simulation of a particle reflecting between two walls. The initial condition is given by

F(s, 0) = 0, F(s, -1) = 0 for s = 1 to s = 99, s = 101 to 200
F(100, 0) = 1, F(100, -1) = 1

In the Figure 6a, the horizontal axis represents the space variable s (from 1 to 200) and the vertical axis represents the time variable t from the top to the bottom. Figures 6bc are the time continuation of Fig. 6a.

These Figures show $P(s, t) = F(s,t)F^*(s,t)$ as a function of space s and time t. The real function is related to the density of presence of the particle between the two walls. The initial particle behaves as a wave spreading until the two walls, reflecting and interfering with a spatial oscillation between the walls. Self-interference is well-seen.

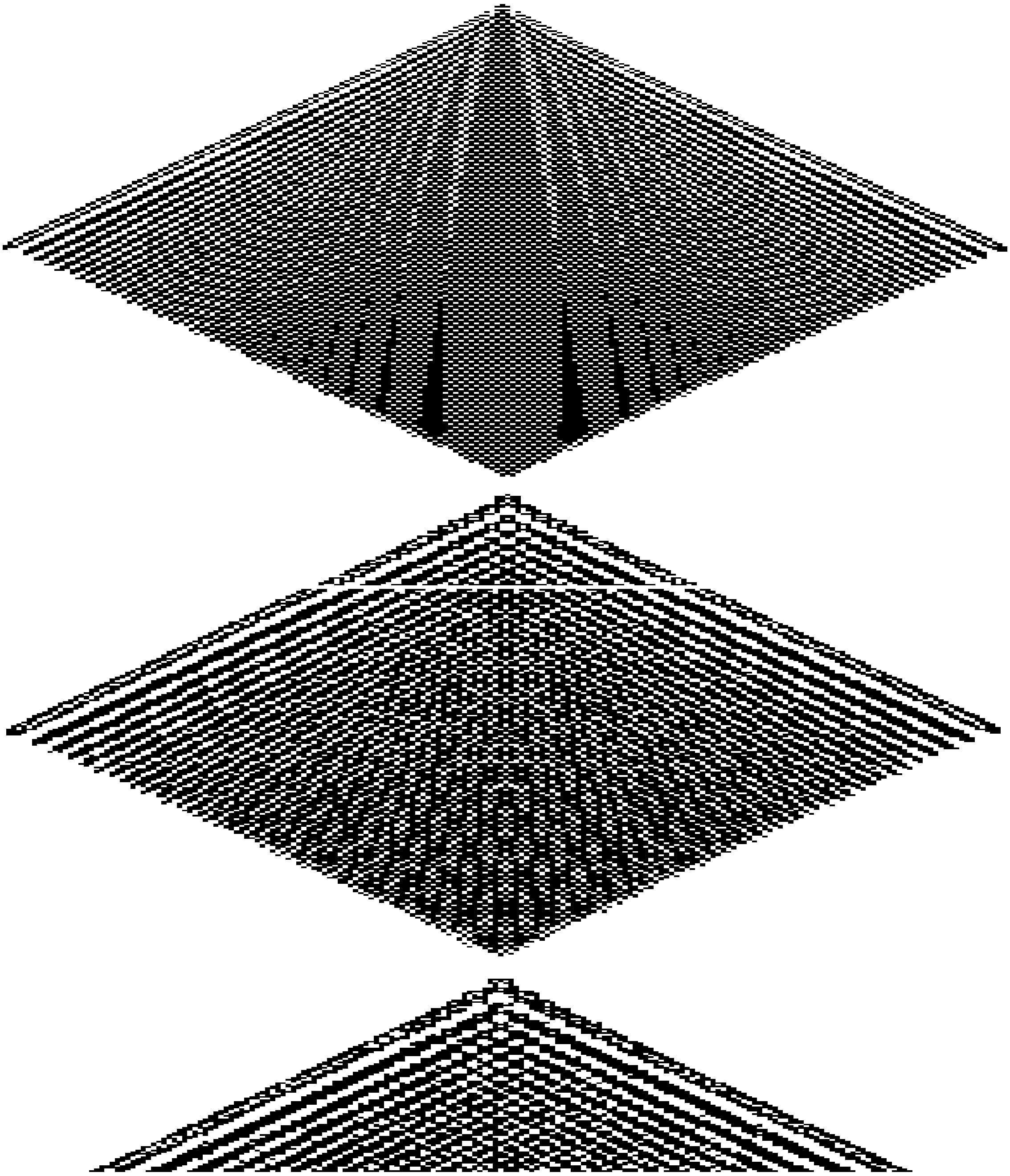
Fig. 7 shows the simulation for two initial particles. The initial conditions are given by

F(s, 0) = 0, F(s, -1) = 0 for s = 1 to s = 49, s = 51 to 149 and s = 151 to 200
F(50, 0) = 1, F(50, -1) = 1
F(150, 0) = 1, F(150, -1) = 1

In the figure 7, the horizontal axis represents the space variable s (from 1 to 200) and the vertical axis represents the time variable t from the top to the bottom.

This Figure shows $P(s, t) = F(s,t)F^*(s,t)$ as a function of space s and time t. The real function is related to the density of presence of the particle between the two walls. The two initial particles behave as waves. They reflect on the walls and interfere: the interferences are well-seen.

The interesting aspect in these two simulations is the fact that the waves record in their interferences an information about the configuration of their space as well as the temporal history of their successive reflections on the walls and their successive interferences. Even a single wave record its self-interferences. So these waves behave like space-time memories. This is in agreement with the path integrals of Feynman.



Figures 6ab

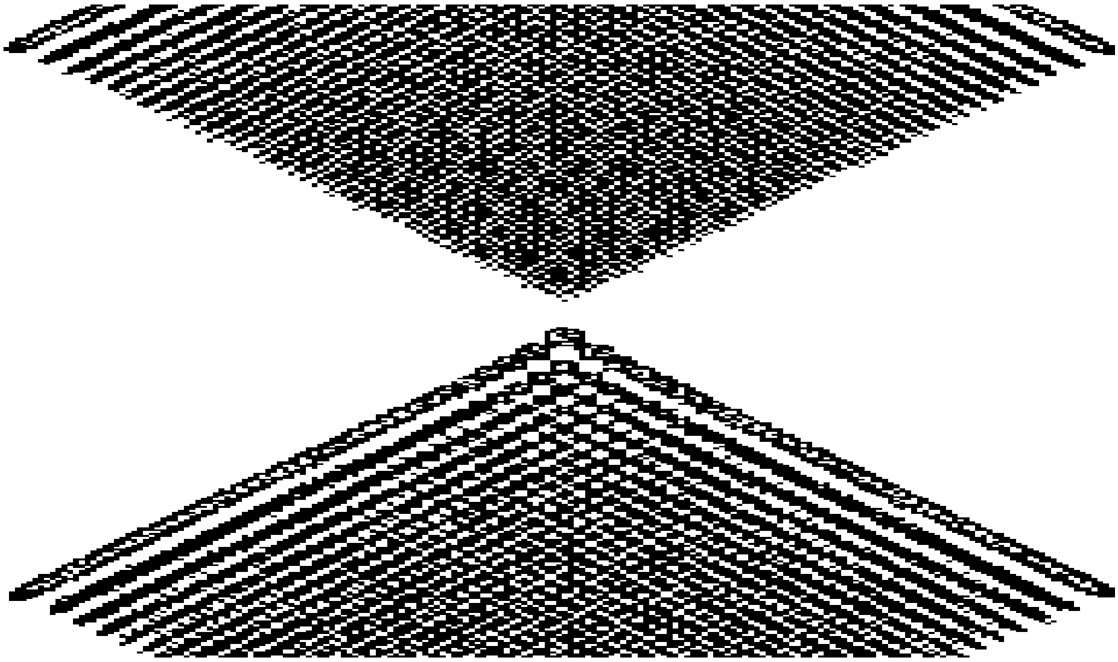


Figure 6c

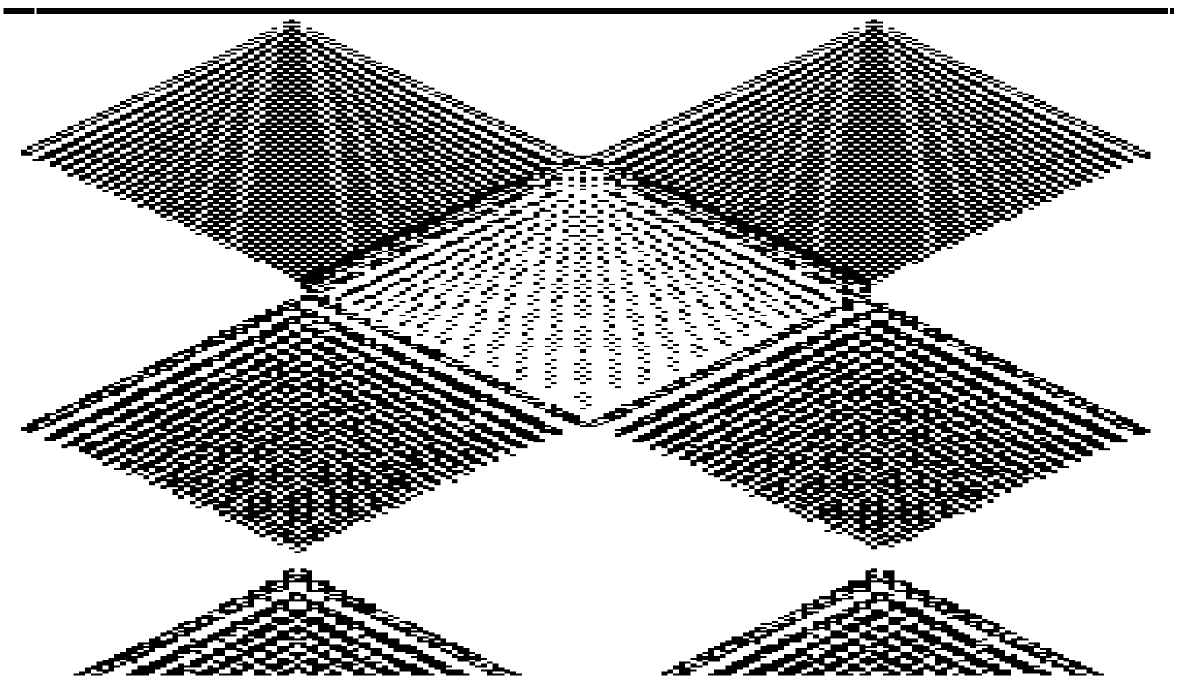


Figure 7

4 Hyperincursive Stack Memory in Chaotic Automata

In this last section of this paper, I present a new model of a neural memory by a stack of binary input data embedded in a floating point variable from an hyperincursive process based on the Pearl-Verhulst chaotic map: $x(t+1)=4\mu x(t)(1-x(t))$.
Theoretical and experimental works enhance the validity of such an approach. Von Neumann (1996) suggests that the brain dynamics is based on hybrid digital-analogical neurons. I proposed a fractal model of neural systems based on the Pearl-Verhulst map (Dubois, 1990, 1992). A non-linear threshold logic was developed from this chaos fractal neuron (Dubois, 1996; Dubois and Resconi, 1993) in relation to the McCulloch and Pitts (1943) formal neuron. Experimental analysis in nervous systems show fractal chaos (King, 1991; Schiff, 1994). Neural systems can be modeled as automata (Weisbuch, 1989). My model of a stack memory could be applied in the framework of symbolic dynamics and coding (Lind, 1995).

The Pearl-Verhulst map in the chaotic zone ($\mu=1$) can be transformed to a quasi-linear map $X(t+1)=1-\mathrm{abs}(1-2x(t))$, where abs means the absolute value. This simple model was proposed for simulating neural chaos (Dubois, 1992). Let us consider the incursive map

$$X(t) = 1 - \mathrm{abs}(1 - 2X(t+1)) \qquad (41)$$

where the iterate X(t) at time t is a function of its iterate at the future time t+1, where t is an internal computational time of the system. Such a relation can be computed in the backward direction T, T-1,T-2, ... 2,1,0 starting with a "final condition" X(T) defined at the future time T, which can be related to the Aristotelian final cause. This map can be transformed to the hyper recursive map (Dubois, 1996c):

$$1 - 2X(t+1) = \pm (1 - X(t)) \quad \text{so} \quad X(t+1) = [1 \pm (X(t) - 1)]/2 \qquad (42)$$

In defining an initial condition X(0), each successive iterates X(t+1), t=0,1,2,...,T
give rise to two iterates due to the double signs $\pm$. So at each step, the number of values increases as 1,2,4,8,... In view of obtaining a single trajectory, at each step, it is necessary to make a choice for the sign. For that, let us define a control function u(T-t) given by a sequence of binary digits 0,1, so that the variable sg

$$sg = 2u(t) - 1 \text{ for } t=1,2,...,T \qquad (43)$$

is -1 for u=0 and +1 for u=1. In replacing eq. 43 in eq. 42, we obtain

$$X(t+1) = [1 + (1 - 2u(t+1))(X(t) - 1)]/2 = X(t)/2 + u(t+1) - X(t).u(t+1) \qquad (44)$$

which is a hyperincursive process because the computation of X(t+1) at time t+1 depends on X(t) at time t and u(t+1) at the future time t+1. Eq. 44 is a soft algebraic map (Dubois, 1996c) generalizing the exclusive OR (XOR) defined in Boolean algebra: $y = x_1 + x_2 - 2x_1x_2$ where x_1 and x_2 are the Boolean

inputs and y the Boolean output. Indeed, in the hybrid system 44, X(t) is a floating point variable and u(t) a digital variable.
Starting with the initial condition X(0)=1/2, this system can memorize any given sequence of any length u. The following Table I gives the successive values of X for all the possible sequences with 3 bits.

TABLE I

u	X(1)	X(1)	X(2)	X(2)	X(3)	X(3)
000	1/4	0.25	1/8	0.125	1/16	0.0625
100	3/4	0.75	3/8	0.375	3/16	0.1875
110	3/4	0.75	5/8	0.625	5/16	0.3125
010	1/4	0.25	7/8	0.875	7/16	0.4375
011	1/4	0.25	7/8	0.875	9/16	0.5625
111	3/4	0.75	5/8	0.625	11/16	0.6875
101	3/4	0.75	3/8	0.375	13/16	0.8125
001	1/4	0.25	1/8	0.125	15/16	0.9375

This table gives the successive values of X for each sequence u as rational and floating point numbers. The number of decimal digits increases in a linear way (one bit of the sequence corresponds to a decimal digit of X). The last digit 5 corresponds to the initial condition X(0)=0.5 and the two last digits 25 or 75 give the parity check of the sequence. The time step t is directly related to the number of digits: with t=0,1,2,3 there are 4 digits. In looking at the successive increasing values of the floating points of X, we see that the correspondent sequences u represent the Gray code. Contrary to the binary code, the Gray code changes only one bit by the unitary addition. The numerator of each ratios is two times the floating point representation of the Gray code of the sequence u, plus one. With the Gray code, we can construct the Hilbert curve which fills the two-dimensions space: the fractal dimension is D_H=2. This is not possible with the Cantor set, which gives discontinuities in two directions in the space (Schroeder, 1991).
The neuron is an analogical device which shows digital spikes: the analogical part of the neuron is given by the floating point values X and the values of the spikes are given by the digital sequence u.
The analogical coding X of digital information of the spikes u is then a learning process which creates a fractal memory.

Now, let us show how it is possible to recover the digital information u(t) for t=,1,2,3,...,T from the analogical final value X(T) of the neuron (T=3 in our example).
The inversion of the sequence has some analogy with the inversion of the image received by the eyes. With our method, the coding of an image leads to an inverse image so that the image is reconstructed without inversion.
The decoding of a sequence u from the final value X(T) can be made by relation 1 for t = T-1, T-2, ...

$$X(t) = 1 - \mathrm{abs}(1 - 2X(t+1)) \qquad (45)$$

Let us take an example, starting with the final value X(T=3)=0.5625, we compute successively X(2) = 1 - abs(1 - 2x0.5625) = 1 - 0.125 = 0.875,
X(1) = 1 - abs(1 - 2x0.875) = 1 - 0.75 = 0.25, X(0) = 1 - abs(1 - 2x0.25) = 0.5.

The sequence is then given by

$$u(t+1) = (2X(t+1)) \text{ div } 1 \quad (46)$$

where div is the integer division: u(3) = (2x0.5625) div 1 = 1, u(2) = 1, u(1) = 0. The neuron will continue to show spikes 1,0,1,0,1,0, ...
It is well-known that neurons are oscillators which present always pulsations, the coding of information is a phase modulation of these pulsations (Dubois, 1992).

In taking the formal neuron of McCulloch and Pitts (1943), eq. 46 can be replaced by

$$u(t+1) = \Gamma\,(X(t+1) - 0.5) \quad (47)$$

for which u=1 if $X \geq 0.5$ and u=0 otherwise.
As we can compute u(t) from X(t), it is possible to compute eq. 45 in the following way

$$X(t) = 2X(t+1) + 2u(t+1) - 4X(t+1)u(t+1) \quad (48)$$

which is also a soft computation of XOR.

So, to retrieve the message embedded in the stack memory by the soft XOR relation 44, a similar soft XOR relation 48 is used.

The following Figure 8a-b gives a possible neural network for the stack memory (Dubois, 1996c).

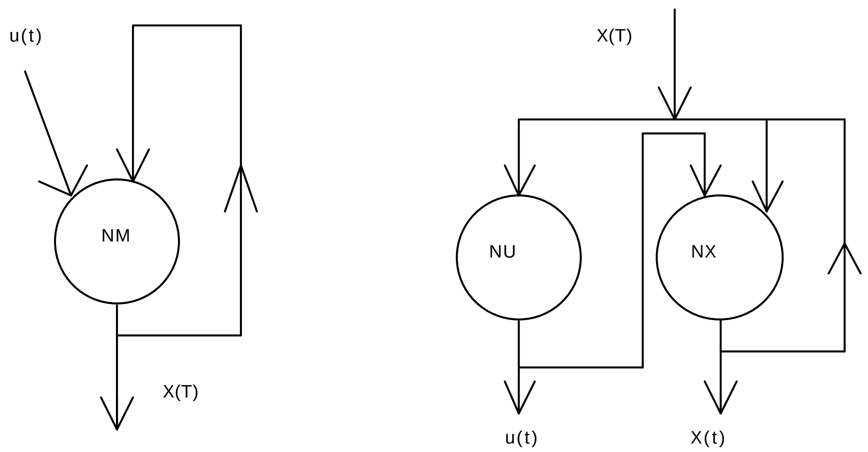


Figure 8a-b: (a) The neuron NM represents the soft XOR eq. 44 for the coding of the sequence u(T-t) giving X(T); (b) The neuron NU is a McCulloch and Pitts neuron given by eq. 47 computing the ordered sequence u(t) and the neuron NX represents the soft XOR eq. 8 giving X(t) starting from the final state X(T) coming from the neuron NM.

This is a property of XOR that the addition and the subtraction is the same operator. Here the soft XOR given by a non-linear algebraic relation gives the same property in a generalized way.

In conclusion, this mast section gives a way to build a stack memory for automata such as neural systems. A hyperincursive control of a fractal chaos map is used for embedding input informations in the state variable of the memory. The input sequence is given by a digital variable and the memory is represented by an analogical variable. The analogical variable is represented in floating point. With classical computer, the number of decimal digit is limited so that we must code the decimal digits of great length by strings. The actual neuron could be an analogical device working only with strings. In this way, such a neural system with hyperincursive stack memory could help in the design of a Hyper Turing Machine.

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